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ON THE fg-COLORING OF GRAPHS

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This paper introduces a new type of edge-coloring of multigraphs, called an fg-coloring, in which each color appears at each vertex v no more than $f(v)$ times and at each set of multiple edges joining vertices v and w no more than $g(vw)$ times. The minimum number of colors needed to fg-color a multigraph G is called the fg-chromatic index of G. Various upper bounds are given on the fg-chromatic index. One of them is a generalization of Vizing's bound for the ordinary chromatic index. Our proof is constructive, and immediately yields a polynomial-time algorithm to fg-color a given multigraph using colors no more than twice the fg-chromatic index.

1. Introduction

In this paper we deal with *multigraphs* which may have multiple edges but have no selfloops; we simply call such a multigraph a *graph*. $G=(V, E)$ denotes a graph with vertex set V and edge set E. We denote by $d(v)$ the degree of vertex v, by $E(vw)$ the set of multiple edges joining vertices v and w, and by $p(vw)$ the cardinality of the set $E(vw)$, that is $p(vw)=|E(vw)|$. An edge in $E(vw$ is) denoted by vw. The *vertex-capacity* f is any function from the vertices V to the natural numbers. The *edge-capacity* g is any function from the edges E to the natural numbers, where it is assumed that $g(e)=g(e')$ for every pair of edges e and e' joining the same two vertices. We define an *fg-coloring* of G as a coloring of the edges in E such that

(a) each vertex v has at most $f(v)$ edges colored with the same color; and
(b) each set $E(vw)$ of multiple edges contains at most $g(vw)$ edges colored with the same color.

The minimum number of colors needed to fg-color a graph G is called the *fg-chromatic index* of G, and is denoted by $q^*_{fg}(G)$.

An ordinary edge-coloring is a special case of an fg-coloring in which $f(v)=1$ for every vertex $v\in V$. An edge-coloring which satisfies (a) but not always (b) is called an *f-coloring* [12] or a "proper edge-coloring" [6, 7], and the minimum number of colors needed to f-color G is called the *f-chromatic index* $q^*_f(G)$ of G. Clearly the f-coloring is also a special case of an fg-coloring in which $g(vw)\geqq\min\{f(v), f(w)\}$ for any $vw\in E$. Several upper bounds on the f-chromatic index have been obtained by Hakimi, Kariv and us [6, 7, 12]. Their "super edge-coloring" [7] is also a special case of an fg-coloring in which $g(vw)=1$ for every $vw\in E$. Hilton and de Werra have

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obtained many notable results on "equitable and edge-balanced colorings" similar to the f- and fg-coloring [8, 17, 18].

One may assume without loss of generality that $g(vw) \leqq \max\{f(v), f(w)\}$ for any edge $vw \in E$. Then in this paper we prove that the following upper bound holds for the fg-chromatic index $q^*_{fg}(G)$:

$$q^*_{fg}(G) \leqq \max_{vw \in E} \lceil d(v)/f(v) + p(vw)/g(vw) \rceil.$$

Throughout the paper $\lceil x \rceil$ means the least integer no less than x, and $\lfloor x \rfloor$ the greatest integer no greater than x. Clearly one may assume that $g(vw) \leqq \min\{f(v), f(w)\}$ but we do not assume so since the bound above increases when g decreases. This upper bound is a generalization of Vizing's bound for the ordinary edge-coloring and Hakimi and Kariv's bound for the f-coloring [7, 15, 16]. The proof is constructive, and immediately yields a polynomial-time algorithm for fg-coloring any given graph with the number of colors assured by the bound. In addition we obtain various upper bounds for $q^*_{fg}(G)$, most of which rather immediately follow from de Werra's results on "equitable and edge-balanced colorings" [17].

2. Preliminaries

In this section we present some notations and two coloring techniques. We denote by $q_{fg}(G)$ the upper bound to be proved, that is

$$q_{fg}(G) = \max_{vw \in E} \lceil d(v)/f(v) + p(vw)/g(vw) \rceil.$$

Let Q be the set of $q_{fg}(G)$ colors available for an fg-coloring of G. An edge colored with color $c \in Q$ is called a *c-edge*. The number of c-edges incident to vertex v is denoted by $d(v, c)$, while the number of c-edges in $E(vw)$ is denoted by $p(vw, c)$. Define $m(v, c) = f(v) - d(v, c)$ and $m(vw, c) = g(vw) - p(vw, c)$. Then G is fg-colored if and only if every color c satisfies $m(v, c) \geqq 0$ for every vertex $v \in V$ and $m(vw, c) \geqq 0$ for every edge $vw \in E$. Color c is *available* at v if $m(v, c) \geqq 1$. Similarly color c is *available* at $E(vw)$ if $m(vw, c) \geqq 1$. We define

$$M(v) = \{c \in Q: m(v, c) \geqq 1\};$$

and

$$M(vw) = \{c \in Q: m(vw, c) \geqq 1\}.$$

Thus $M(v)$ is the set of colors available at vertex v, while $M(vw)$ is the set of colors available at multiple edges $E(vw)$.

Switching an alternating path is one of the standard techniques of an ordinary or f-coloring [3, 6, 7, 12, 15, 16, 17, 18]. We also use it with some modifications. A walk is used instead of a path. A *walk* W is a sequence of distinct edges v_0v_1, $v_1v_2, \ldots, v_{k-1}v_k$, where the vertices $v_0, v_1, \ldots, v_k$ are not necessarily distinct. The *length* of W is the number of edges in W. Vertex v_0 is the *start vertex* of W and v_k the *end vertex*. Walk W is called a *cycle* if $v_0 = v_k$. When the edges of a walk W are colored with two colors a and b alternately, *switching* W means to interchange the colors a and b of the edges in W. We define an "ab-alternating walk" so that its switch would preserve an fg-coloring of G, as follows.

Let $G(a, b)$ be the subgraph of G induced by all a- and b-edges. Delete successively all pairs of edges of color a and b respectively joining the same two vertices, and let $G^*(a, b)$ be the resulting graph in which there no longer exists such a pair. Denote by $E^*(vw)$ the set of multiple edges joining vertices v and w in $G^*(a, b)$. Obviously each set $E^*(vw)$ contains only a- or b-edges. An *ab-alternating walk* $W = v_0v_1, v_1v_2, \ldots, v_{k-1}v_k$ is a walk of length one or more in $G^*(a, b)$ such that

(i) the edges in W are colored alternately with a and b (the ith edge e_i of W is colored b if i is an odd number; otherwise e is colored a);

(ii) if W is not a cycle then $m(v_0, a) \geqq 1$, while if W is a cycle of odd length then $m(v_0, a) \geqq 2$; and

(iii) if W is not a cycle and is of even length then $m(v_k, b) \geqq 1$, while if W is not a cycle and is of odd length then $m(v_k, a) \geqq 1$.

Thus a cycle in $G^*(a, b)$ is an ab-alternating walk (cycle) whenever it has even length and its edges are colored a and b alternately. The following lemma holds.

Lemma 1. *Let G be fg-colored, and let W be any ab-alternating walk in G. Furthermore, let m' represent the function m with respect to the new coloring after switching W. Then the following (a)–(c) hold.*

(a) Switching W preserves an fg-coloring of G, that is, every color c satisfies $m'(v, c) \geqq 0$ for any $v \in V$ and $m'(vw, c) \geqq 0$ for any $vw \in E$.

(b) For any $vw \in E$

$$m'(vw, a) \geqq \min\{m(vw, a), m(vw, b)\},$$

and

$$m'(vw, b) \geqq \min\{m(vw, a), m(vw, b)\}.$$

(c) If W passes through an a-edge in $E(vw)$ then $m'(vw, a) > 0$, while if W passes through a b-edge in $E(vw)$ then $m'(vw, b) > 0$.

Proof. (a) Clearly switching W does not change $m(v, c)$ or $m(vw, c)$ for any color $c \neq a, b$. Also switching W does not change $m(vw, a)$ and $m(vw, b)$ unless W passes through an edge in $E(vw)$. Thus we shall verify that $m'(v, a) \geqq 0$ and $m'(v, b) \geqq 0$ for every vertex v on W and that $m'(vw, a) \geqq 0$ and $m'(vw, b) \geqq 0$ for every edge vw on W.

Switching W may change $m(v, c)$, $c = a$ or b, only at the start vertex v_0 or the end vertex v_k of W. However conditions (ii) and (iii) above implies that after switching W, $m'(v, c) \geqq 0$ for $c = a, b$ and $v = v_0, v_k$.

Let vw be an edge in W. One may assume without loss of generality that vw is an a-edge and hence $E^*(vw)$ contains only a-edges. Then

$$m(vw, b) = m(vw, a) + |E^*(vw)|;$$

and

$$m(vw, a) = \min\{m(vw, a), m(vw, b)\}.$$

Since switching W decreases the number of a-edges in $E(vw)$ at least one,

$$m'(vw, a) > m(vw, a) \geqq 0.$$

Since switching W increases the number of b-edges in $E(vw)$ at most $|E^*(vw)|$,

$$m'(vw, b) \geqq m(vw, b) - |E^*(vw)| = m(vw, a) \geqq 0.$$

This completes the proof of (a). The proof of (b) and (c) is also implicit in the proof above. ▌

Lemma 1(b) implies that if $m(vw, a)$, $m(vw, b) \geqq 1$ then $m'(vw, a)$, $m'(vw, b) \geqq 1$ after switching any ab-alternating walk.

We denote by $W(a, b, v_0)$ an ab-alternating walk which starts with vertex v_0 and is not a cycle of even length. Swithing $W(a, b, v_0)$ may change $m(v, a)$ or $m(v, b)$ only if v is the start or end vertex. On the other hand, switching an ab-alternating cycle of even length changes neither $m(v, a)$ nor $m(v, b)$ for any $v \in V$.

Lemma 2. *Let $v_0 \in V$, $a, b \in Q$, and $a \in M(v_0)$ in an fg-coloring of a graph $G=(V, E)$. Then the following (a) and (b) hold.*

(a) *If $G^*(a, b)$ has a b-edge e_1 incident to vertex v_0, then G has an ab-alternating walk W starting with v_0 and passing through e_1.*

(b) *If $b \notin M(v_0)$, then G has a walk $W(a, b, v_0)$.*

Proof. (a) Note first that for every vertex v the difference $d(v, a) - d(v, b)$ with respect to G is the same as that with respect to $G^*(a, b)$. Then one can construct an ab-alternating walk W as follows. Choose the b-edge $e_1 \in E(v_0 v_1)$ as the first edge of W. If $m(v_1, a) \geqq 1$ then the single edge e_1 is an ab-alternating walk. So suppose that $m(v_1, a) = 0$. Then $d(v_1, a) - d(v_1, b) \geqq 0$, and consequently $G^*(a, b)$ has an a-edge $e_2 \in E^*(v_1 v_2)$ incident to v_1. Add the a-edge e_2 to W as the second edge. Similarly repeat adding an edge to W, choosing alternately a- and b-edges which have not been included in W so far, until the conditions (ii) and (iii) above are satisfied for the start and end vertices of W.

Especially when W returns to the start vertex v_0, we proceed the construction of W as follows. If W returns to v_0 with an a-edge, then end the construction of W. In this case an ab-alternating cycle W of even length is obtained. Also if W returns to v_0 with a b-edge and $m(v_0, a) \geqq 2$, then end the construction of W. In this case an ab-alternating cycle of odd length is obtained. If W returns to v_0 with a b-edge but $m(v_0, a) = 1$, then add to W an a-edge incident to v_0 and continue the construction of W; since $d(v_0, a) - d(v_0, b) \geqq -1$, $G^*(a, b)$ contains such an a-edge which has not been included in W so far.

(b) Since $m(v_0, a) \geqq 1$ and $m(v_0, b) = 0$, $d(v_0, b) - d(v_0, a) \geqq 1$ and hence $G^*(a, b)$ has a b-edge incident to v_0. Therefore (a) above implies that there is an ab-alternating walk W starting with v_0. Since $d(v_0, b) - d(v_0, a) \geqq 1$, one can choose a b-edge incident to v_0 which has not been included in W so far whenever W returns to v_0 with an a-edge. Thus one can construct an ab-alternating walk $W(a, b, v_0)$ which is not a cycle of even length. ▌

In the case of an ordinary or f-coloring, if the ends of an uncolored edge vw have a common available color c, that is, $c \in M(v) \cap M(w)$, then the coloring of G proceeds with coloring vw c. It is not the case when fg-coloring a graph. For, if $m(vw, c) = 0$, then one cannot color an uncolored edge vw with a color $c \in M(v) \cap M(w)$. Thus an uncolored edge $e \in E(vw)$ can be colored with color c only if $c \in M(v) \cap M(w) \cap M(vw)$. Such a color does not always exist in a partial fg-coloring of G. [We will later show that any partial fg-coloring of G using $q_{fg}(G)$ colors can always be altered so that there is a color $c \in M(v) \cap M(w) \cap M(vw)$ for the ends of an uncolored edge vw.] However, by a simple counting argument one can show that both $M(v) \cap M(vw)$ and $M(w) \cap M(vw)$ contain at least one color, as follows.

Lemma 3. *Let G be fg-colored with at most $q_{fg}(G)$ colors in Q. Then for any edge $vw \in E$*

$$\sum_{c \in M(vw)} m(v, c) \geqq 1. \tag{1}$$

Especially if $g(vw) \leqq f(v)$, *then*

$$\sum_{c \in M(vw)} m(v, c) \geqq p(vw). \tag{2}$$

Proof. By the definition of $q_{fg}(G)$

$$q_{fg}(G) \geqq d(v)/f(v) + p(vw)/g(vw). \tag{3}$$

Let S be the set of colors not available at $E(vw)$, that is,

$$S = \{c \in Q\colon\ m(vw, c) = 0\} \quad (= Q - M(vw)).$$

Let t be the number of edges in $E(vw)$ which are colored with colors in S, that is

$$t = |S|\, g(vw). \tag{4}$$

Then we have

$$\sum_{c \in M(vw)} m(v, c) = \sum_{c \in M(vw)} \{f(v) - d(v, c)\} \geqq \tag{5}$$

$$\geqq f(v)\{q_{fg}(G) - |S|\} - \{d(v) - t\}.$$

From equations (3), (4) and (5) we have

$$\sum_{c \in M(vw)} m(v, c) \geqq d(v) + f(v)p(vw)/g(vw) - f(v)t/g(vw) - d(v) + t =$$

$$= \big(p(vw) - t\big)f(v)/g(vw) + t.$$

Noting the fact that $p(vw) - t \geqq 0$, one can easily verify equations (1) and (2) from the equation above. ∎

Note that if $g(vw) \leqq \max\{f(v), f(w)\}$ then either

$$\sum_{c \in M(vw)} m(v, c) \geqq p(vw) \quad \text{or} \quad \sum_{c \in M(vw)} m(w, c) \geqq p(vw).$$

"Shifting a fan" is another standard technique of an ordinary edge coloring. For example, it is employed in the proof of Vizing's theorem [3, 15, 16]. We also use it with modifying the definition of a fan suitable for an *fg*-coloring. Let $e_0 = wv_0$ be an uncolored edge. Then a *fan F* is a sequence of distinct edges $e_0, e_1, \ldots, e_k$ incident to vertex $w \in V$ such that there is a sequence of distinct colors $a_0, a_1, \ldots, a_{k-1}$ satisfying the following conditions (a) and (b), where v_i, $0 \leqq i \leqq k$, is the end of e_i other than w:

(a) $\quad a_i \in M(v_i) \cap M(wv_i), \quad 0 \leqq i \leqq k-1;$

and

(b) $\quad e_i, \quad 1 \leqq i \leqq k, \quad$ is colored with a_{i-1}.

A fan is illustrated in Fig. 1. Note that vertices $v_0, v_1, v_2, \ldots, v_k$ are not always distinct. *Shifiting a fan F* means to recolor e_i with a_i for each i, $0 \leqq i \leqq k-1$, and erase the color a_{k-1} of e_k. Shifting F yields another fg-coloring of G in which e_k instead of e_0 is uncolored.

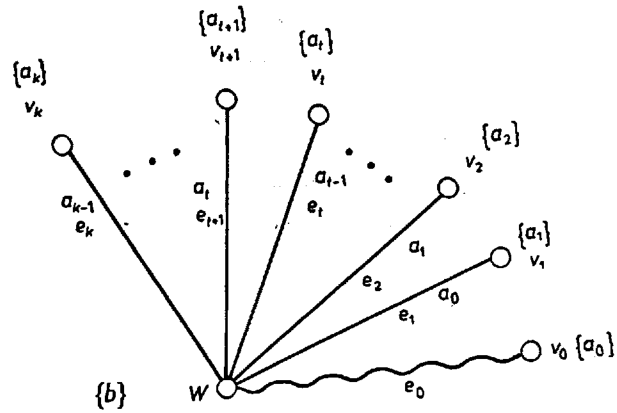

Fig. 1. Fan

3. Main theorem

The following theorem is a main result of this paper.

Theorem 4. *Assume without loss of generality that* $g(vw) \leqq \max\{f(v), f(w)\}$ *for any edge* $vw \in E$ *of a graph* $G=(V, E)$. *Then*

$$q^*_{fg}(G) \leqq \max_{vw \in E} \lceil d(v)/f(v) + p(vw)/g(vw) \rceil.$$

Proof. We will prove the claim by induction on the number of edges. Let w be a vertex of degree one or more having the minimum vertex-capacity $f(w)$ among such vertices. Let $e_0 \in E(vw)$. By the inductive hypothesis the graph $G-e_0$ obtained from G by deleting e_0 can be fg-colored with $q_{fg}(G)$ $(\geqq q_{fg}(G-e_0))$ colors. We shall prove that G also can be fg-colored with $q_{fg}(G)$ colors.

It follows from the selection of w that $g(wv) \leqq f(v)$ for any vertex v adjacent to w. Therefore, by Lemma 3, for any vertex v adjacent to w

$$\sum_{a \in M(wv)} m(v, a) \geqq p(wv),$$

and

$$\sum_{a \in M(wv)} m(w, a) \geqq 1.$$

Noting this fact, we construct a fan F as follows. Clearly a single edge e_0 is a fan. Assume in general that fan $F=e_0, e_1, \ldots, e_k$ has been constructed so far. If there exists a color $a_k \in M(wv_k)$ such that $m(w, a_k)+m(v_k, a_k) \geqq 2$, then stop the construction of F here and let $F=e_0, e_1, \ldots, e_k$. Otherwise, there exists a color $a_k \in M(wv_k)$ such that

(1) $m(v_k, a_k) = 1$;

(2) $m(w, a_k) = 0$;

and

(3) if $v_i = v_k$, $0 \leqq i \leqq k-1$, then $a_i \neq a_k$.

This follows from

$$\sum_{a \in M(wv_k)} m(v_k, a) \geqq p(wv_k).$$

If a_k is the same as $a_0, a_1, \ldots, a_{k-1}$, then stop the construction of F here, and let $F=e_0, e_1, \ldots, e_k$. Otherwise, continue the construction of F with adding to F an a_k-edge $e_{k+1} \in E(wv_{k+1})$ incident to w and adding the color a_k to the color sequence.

Let $F=e_0, e_1, e_2, \ldots, e_k$ be the fan F constructed as above. Then $a_k \in M(wv_k)$, and one of the following three cases happens.

Case 1. $m(w, a_k) \geqq 1$ and $m(v_k, a_k) \geqq 1$.

In this case $a_k \in M(w) \cap M(v_k) \cap M(wv_k)$. Furthermore $v_i \neq v_k$, $0 \leqq i \leqq k-1$, since there is no color $a_i \in M(wv_i)$ such that $m(w, a_i)+m(v_i, a_i) \geqq 2$. After shifting F, $a_k \in M(w) \cap M(v_k) \cap M(wv_k)$ as it was. Therefore, shifting F and coloring the new uncolored edge e_k with a_k completes an *fg*-coloring of G.

Case 2. $m(v_k, a_k) \geqq 2$ or $m(w, a_k) \geqq 2$.

Also in this case $v_i \neq v_k$, $0 \leqq i \leqq k-1$. Assume $m(v_k, a_k) \geqq 2$. (The proof for the case $m(w, a_k) \geqq 2$ is similar.) By Lemma 3 there is a color $b \in M(wv_k)$ such that $m(w, b) \geqq 1$. One may assume that Case 1 does not apply. Then $m(v_k, b)=0$ and $b \neq a_{k-1}$. Shift fan F, then $e_k=wv_k$ becomes the new uncolored edge, and $a_k, b \in M(wv_k)$, $a_k \in M(v_k)$ and $b \notin M(v_k)$. By Lemma 2(b) there exists an $a_k b$-alternating walk $W(a_k, b, v_k)$ which starts with v_k and is not a cycle of even length. Switch $W(a_k, b, v_k)$, then by Lemma 1(b) $a_k, b \in M(wv_k)$ after switching it since a_k, $b \in M(wv_k)$ before switching it. If $W(a_k, b, v_k)$ ended at w, then $a_k \in M(w) \cap M(v_k) \cap$ $\cap M(wv_k)$ and consequently coloring the uncolored edge $e_k=wv_k$ with color a_k completes an *fg*-coloring of G. If $W(a_k, b, v_k)$ did not end at w, then $b \in M(w) \cap$ $\cap M(v_k) \cap M(wv_k)$ and coloring e_k with b completes an *fg*-coloring of G.

Case 3. $m(v_k, a_k)=1$, $m(w, a_k)=0$, and $a_k=a_t$ for some t, $0 \leqq t < k$.

One may assume that neither Case 1 nor Case 2 applies. By (3) above $v_t \neq v_k$. (See Fig. 1). By Lemma 3 there exists a color $b \in M(w) \cap M(wv_t)$. If $b \notin M(v_k)$, then by Lemma 2(b) there is an $a_k b$-alternating walk $W(a_k, b, v_k)$ which is not a cycle of even length. If $b \in M(v_k)$, then $b \notin M(wv_k)$ since Case 1 did not apply, and consequently $G^*(a_k, b)$ contains a b-edge $e' \in E^*(wv_k)$. Therefore by Lemma 2(a) there is an $a_k b$-alternating walk W starting with v_k and passing through e'. Since $m(v_k, a_k)=1$, W is not a cycle of odd length, but may be a cycle of even length. Thus, no matter whether $b \in M(v_k)$ or not, there is an $a_k b$-alternating walk W starting with v_k. Construct such a walk W in the following way:

(a) If $b \notin M(wv_k)$, then choose a b-edge in $E^*(wv_k)$ as the first edge of W;

(b) If W reaches w with an a_k-edge, then terminate W there; and

(c) If W reaches w with a b-edge, then add to W an a_k-edge other than e_{t+1} which is incident to w and has not been included in W (there is such an edge since $m(w, a_k)=0$ and $m(w, b)=1$ and consequently $d(w, a_k)> >d(w, b)$).

The condition (a) above together with Lemmas 1(b) and (c) imply that switching W makes $b \in M(wv_k)$ no matter whether $b \in M(wv_k)$ or not before switching W. Here $b \neq a_0, a_1, \ldots, a_{k-1}$ because $m(w, b)=1$ and $m(w, a_i)=0$, $0 \leq i \leq k-1$. Since F is a fan, colors $a_0, a_1, \ldots, a_{k-1}$ are all distinct. Therefore W does not pass through any edges of F except the a_k-edge e_{t+1}. Furthermore (b) and (c) above imply that if W passes thorugh the a_k-edge e_{t+1}, then W must end at w. We separate this case into three subcases depending on the end vertex of W.

Case 3.1: W ends at neither v_t nor w.

In this case W passes through none of the edges of F. Switch W, then $m(wv_k, b) \geqq 1$, and $m(w, b) \geqq 1$ as it was. Furthermore we claim that F remains to be a fan as it was. Since W passes through none of the edges of F, switching W does not change the coloring of the edges in F at all. Although W may pass through edges in $E(wv_i)$, $1 \leqq i \leqq k-1$, switching W clearly preserves $a_i \in M(v_i) \cap M(wv_i)$ for each i, $0 \leqq i \leqq k-1$, except t since colors $b, a_0, a_1, \ldots, a_{k-1}$ are distinct and $a_i \neq a_k$ if $i \neq t$. On the other hand, since $a_t, b \in M(wv_t)$ before switching W, by Lemma 1(b) $a_t, b \in M(wv_t)$ after switching W. Furthermore $a_t \in M(v_t)$ as it was since W did not end at v_t. Hence $a_t \in M(v_t) \cap M(wv_t)$ as it was. Thus we have shown that F remains to be a fan as it was.

If W did not end at v_k, that is, W was not a cycle of even length, switching W makes $b \in M(v_k)$. If W ended at v_k, that is, W was a cycle of even length, then $b \in M(v_k)$ after switching W since $b \in M(v_k)$ before switching W. Thus in either case $b \in M(w) \cap \cap M(v_k) \cap M(wv_k)$ after switching W. Therefore shifting F and coloring the new uncolored edge $e_k = wv_k$ with b complete an fg-coloring of G.

Case 3.2: W ends at v_t.

W passes through none of the edges of F. Switch W, then $m(v_t, b) \geqq 1$, and $m(w, b) \geqq 1$ and $a_t, b \in M(wv_t)$ as it was. Since colors $b, a_0, a_1, \ldots, a_{k-1}$ are all distinct, switching W does not destroy the subfan $e_0, e_1, \ldots, e_t$ of F. Shift the subfan, then $e_t = wv_t$ becomes the new uncolored edge and $b \in M(w) \cap M(v_t) \cap M(wv_t)$. Therefore coloring e_t with b completes an fg-coloring of G.

Case 3.3: W ends at w.

Since $m(w, a_k)=m(v_t, b)=0$ and $m(w, b)=m(v_t, a_t)=1$, $d(v_t, a_t)<d(v_t, b)$ and $d(w, a_t)=d(w, b)$ in the graph $G^*(a_k, b)-W$ obtained from $G^*(a_k, b)$ by deleting all the edges of W. Therefore there exists $W(a_k, b, v_t)$ which does not end at w and is edge-disjoint with W. Switching $W(a_k, b, v_t)$ makes $b \in M(w) \cap M(v_t) \cap \cap M(wv_t)$ and does not destroy the subfan $e_0, e_1, \ldots, e_t$ of F. Shifting the subfan and coloring the new uncolored edge e_t with b completes an fg-coloring of G. ▮

Hakimi and Kariv's upper bound on an f-coloring [7, Theorem 3] and Vizing's theorem for an ordinary edge-coloring immediately follow from Theorem 4 as a corollary.

Corollary 5. *The f-chromatic index $q_f^*(G)$ of a graph G satisfies*

$$q_f^*(G) \leqq \max_{vw \in E} \lceil (d(v)+p(vw))/f(v) \rceil.$$

Proof. Define an edge-capacity g in terms of the given vertex-capacity f as follows:

$$g(vw) = \max\{f(v), f(w)\} \quad \text{for every} \quad vw \in E. \quad \blacksquare$$

Corollary 6. [15, 16] *Let $q^*(G)$ be the chromatic index of a graph G, that is the minimum number of colors required for an ordinary edge-coloring of G. Then*

$$q^*(G) \leqq \max_{vw\in E}\{d(v)+p(vw)\}. \quad \blacksquare$$

Corollary 7. *For a positive integer k, let $q^*_{fk}(G)$ be the fg-chromatic index $q^*_{fg}(G)$ of a graph G for the case in which $g(vw)=k$ for any $vw\in E$. If $k \leqq \min_{vw\in E}\max\{f(v), f(w)\}$, then*

$$q^*_{fk}(G) \leqq \max_{vw\in E}\lceil d(v)/f(v)+p(vw)/k\rceil.$$

Especially every graph G satisfies

$$q^*_{f1}(G) \leqq \max_{vw\in E}\{\lceil d(v)/f(v)\rceil + p(vw)\}. \quad \blacksquare$$

Although Hakimi and Kariv obtained another upper bound on $q^*_{f1}(G)$ [7, Theorem 7], there is no implication between ours and theirs.

4. Miscellaneous results

Hilton and de Werra have investigated an "equitable edge-balanced coloring" in which edges incident to each vertex and each collection of multiple edges are colored equitably in number [8, 17, 18]. One can derive Theorems 8, 9 and 10 below from their results. Firstly we show that the fg-chromatic index of a bipartite graph can be easily decided, as follows.

Theorem 8. *The fg-chromatic index $q^*_{fg}(G)$ of a bipartite graph G satisfies the following equation:*

$$q^*_{fg}(G) = \max\left\{\max_{v\in V}\lceil d(v)/f(v)\rceil,\ \max_{vw\in E}\lceil p(vw)/g(vw)\rceil\right\}.$$

Proof. de Werra [17, Corollary 2.1.1] has shown that for any positive integer q the edges of a bipartite graph $G=(V, E)$ can be colored with q colors (that is, E is partitioned into q subsets) such that any colors a and b satisfy

$$|d(v, a)-d(v, b)| \leqq 1 \quad \text{for each } v\in V; \text{ and}$$

$$|p(vw, a)-p(vw, b)| \leqq 1 \quad \text{for each } vw\in E.$$

Choose q as

$$q = \max\left\{\max_{v\in V}\lceil d(v)/f(v)\rceil,\ \max_{vw\in E}\lceil p(vw)/g(vw)\rceil\right\}.$$

Since clearly $q \leqq q^*_{fg}(G)$, we shall prove $q^*_{fg}(G) \leqq q$. It suffices to verify that the coloring above using q colors is indeed an fg-coloring. Suppose to the contrary that $d(v, a) \geqq f(v)+1$ for some vertex v and color a. Then every color b other than a satisfies $d(v, b) \geqq f(v)$, and consequently

$$d(v) \geqq f(v)+1+(q-1)f(v) = qf(v)+1.$$

This contradicts the selection of q. Thus $d(v, a) \leqq f(v)$ for any vertex v and color a. Similarly one can easily show that $p(vw, a) \leqq g(vw)$ for any edge $vw \in E$ and any color a. Hence the coloring with q colors is an fg-coloring. ∎

The following theorem represents an upper bound on the fg-chromatic index in terms of the f-chromatic index.

Theorem 9. *If $g(vw) \geqq 2$ for every edge $vw \in E$ of a graph G, then*

$$q_{fg}^*(G) \leqq \max \{q_f^*(G), \max_{vw \in E} \lceil (p(vw)-1)/(g(vw)-1) \rceil\}.$$

Proof. de Werra [17, Theorem 2.2] has shown that for any set Q of q colors an arbitrary edge-coloring of a graph $G=(V, E)$ with q colors (that is, an arbitrary partition of E into q subsets) can be altered so that

$$|p'(vw, a)-p'(vw, b)| \leqq 2 \quad \text{for any} \quad vw \in E \quad \text{and} \quad a, b \in Q; \quad \text{and}$$

$$\max_{a \in Q} d'(v, a) \leqq \max_{a \in Q} d(v, a) \quad \text{for any vertex} \quad v \in V,$$

where d and p represent functions with respect to the coloring before the alteration and d' and p' represent functions after the alteration. Choose q as

$$q = \max \{q_f^*(G), \max_{vw \in E} \lceil (p(vw)-1)/(g(vw)-1) \rceil\},$$

and obtain an f-coloring of G with q colors. Alter the f-coloring as above. Then we claim that the resulting coloring is an fg-coloring.

Since $d(v, a) \leqq f(v)$ for any vertex $v \in V$ and color $a \in Q$, $d'(v, a) \leqq f(v)$ for any $v \in V$ and color $a \in Q$.

Suppose that $p'(vw, a) \geqq g(vw)+1$ for some edge $vw \in E$ and color $a \in Q$. Then $p'(vw, b) \geqq g(vw)-1$ for any color b other than a, and consequently

$$p(vw) \geqq g(vw)+1+(g(vw)-1)(q-1) = q(g(vw)-1)+2.$$

This contradicts the selection of q. Thus $p'(vw, a) \leqq g(vw)$ for any $vw \in E$ and $a \in Q$. Thus the altered coloring is an fg-coloring. ∎

Furthermore we have:

Theorem 10. *If $f(v) \geqq 2$ for every vertex $v \in V$ of a graph $G=(V, E)$, then*

$$q_{fg}^*(G) \leqq \max \{\max_{v \in V} \lceil (d(v)-1)/(f(v)-1) \rceil, \max_{vw \in E} \lceil p(vw)/g(vw) \rceil\}.$$

Proof. de Werra [17, Theorem 2.3] has shown that for any positive integer q the edges of any graph G can be colored with q colors so that any colors a and b satisfy

$$|d(v, a)-d(v, b)| \leqq 2 \quad \text{for any vertex} \quad v \in V,;$$

and

$$|p(vw, a)-p(vw, b)| \leqq 1 \quad \text{for any edge} \quad vw \in E.$$

Choose q as

$$q = \max \{\max_{v \in V} \lceil (d(v)-1)/(f(v)-1) \rceil, \max_{vw \in E} \lceil p(vw)/g(vw) \rceil\}.$$

Then one can easily show that the coloring above is indeed an fg-coloring. ∎

There is no implication among Theorems 4, 9 and 10. The following theorem implies Hakimi and Kariv's bound for the "super coloring" [7, Theorem 8].

Theorem 11. *Replace each set $E(vw)$ of multiple edges in a graph G with a set of $g(vw)$ multiple edges, and let G' be the resulting graph. Furthermore let* $s=\max_{vw\in E}\lceil p(vw)/g(vw)\rceil$. *Then*

$$q^*_{fg}(G) \leqq sq^*_f(G').$$

Corollary 12. [7]. *Let G' be the underlying simple graph of a graph G, that is, G' is a simple graph obtained from G by replacing each set of multiple edges with a single edge. Then*

$$q^*_{f1}(G) \leqq (\max_{vw\in E} p(vw))q^*(G').$$

We can derive the following theorem from Theorem 8.

Theorem 13. *If* $f(v)\geqq 2$ *for any* $v\in V$ *and* $g(vw)\geqq 2$ *for any* $vw\in E$ *in a graph* $G=(V, E)$, *then*

$$q^*_{fq}(G) \leqq \max_{vw\in E} \{\lceil\lfloor d(v)/2\rfloor/\lfloor f(v)/2\rfloor\rceil, \lceil\lceil d(v)/2\rceil/\lceil f(v)/2\rceil\rceil,$$

$$\lceil\lfloor p(vw)/2\rfloor/\lfloor g(vw)/2\rfloor\rceil, \lceil\lceil p(vw)/2\rceil/\lceil g(vw)/2\rceil\rceil\}.$$

Proof. $E^+(vw)$ denotes the set of multiple edges going from vertex v to w in a directed graph, while $E^-(vw)$ denotes the set of multiple edges going from w to v. Define

$$p^+(vw) = |E^+(vw)|;$$

$$p^-(vw) = |E^-(vw)|;$$

$$d^+(v) = \sum_{w\in V} p^+(vw);$$

and

$$d^-(v) = \sum_{w\in V} p^-(vw).$$

Appropriately directing the edges of a given undirected graph $G=(V, E)$, we can obtain a directed graph $G_1=(V, E_1)$ such that

(a) $|p^+(vw)-p^-(vw)| \leqq 1$ for any edge $vw\in E$ of G_1;

and

(b) $|d^+(v)-d^-(v)| \leqq 1$ for any vertex $v\in V$ of G_1.

Direct a pair of undirected multiple edges in G in the two opposite directions whenever there is such a pair in G. Let G' be the subgraph of G induced by the edges which have not been directed so far, then clearly G' is a simple graph. Add a new vertex x to G', and join x to each vertex of odd degree in G'. Let G'' be the resulting Eulerian graph. Direct the remaining edges of G along an Eulerian cycle of G''. Then a desired directed graph G_1 is obtained.

Construct an undirected graph $G_2=(V_2, E_2)$ from the directed graph G_1 as follows. Associate two vertices v^+ and v^- with each vertex $v\in V$, and let $V_2=\{v^+,$

v^-: $v \in V\}$. For each set $E^+(vw)$ of multiple directed edges going from v to w in G_1, add to G_2 $p^+(vw)$ multiple undirected edges joining v^+ and w^-. Thus G_2 is an undirected bipartite graph, and there is a one to one correspondence between the edges of G_2 and the edges of G.

Define the vertex-capacity function f_2 and edge-capacity function g_2 for G_2 in terms of f and g for G as follows:

(a) if $d(v^+) \leqq d(v^-)$ in G_2, then

$$f_2(v^+) = \lfloor f(v)/2 \rfloor \quad \text{and} \quad f_2(v^-) = \lceil f(v)/2 \rceil;$$

(b) if $d(v^+) = d(v^-)+1$ in G_2, then

$$f_2(v^+) = \lceil f(v)/2 \rceil \quad \text{and} \quad f_2(v^-) = \lfloor f(v)/2 \rfloor;$$

(c) if $p(v^+w^-) \leqq p(v^-w^+)$ in G_2, then

$$g_2(v^+w^-) = \lfloor g(vw)/2 \rfloor \quad \text{and} \quad g_2(v^-w^+) = \lceil g(vw)/2 \rceil;$$

and

(d) if $p(v^+w^-) = p(v^-w^+)+1$ in G_2, then

$$g_2(v^+w^-) = \lceil g(vw)/2 \rceil \quad \text{and} \quad g_2(v^-w^+) = \lfloor g(vw)/2 \rfloor.$$

Denote by $d_2(v)$ the degree of a vertex $v \in V_2$ in G_2, and by $p_2(vw)$ the number of multiple edges joining vertices v and w. Then by Theorem 8 the f_2g_2-chromatic index of the bipartite graph G_2 is

$$q^*_{f_2g_2}(G_2) = \max\left\{\max_{v \in V_2} \lceil d_2(v)/f_2(v) \rceil, \max_{vw \in E_2} \lceil p_2(vw)/g_2(vw) \rceil\right\}.$$

From an f_2g_2-coloring of G_2 using $q^*_{f_2g_2}(G_2)$ colors one can obtain an fg-coloring of G using the same number of colors. ∎

From Theorem 13 one can easily obtain the following corollary which is similar to Shannon's bound [14] for an ordinary edge coloring.

Corollary 14. *Assume that $f(v) \geqq 2$ for any $v \in V$ and $g(vw) \geqq 2$ for any $vw \in E$ in a graph G, and let*

$$d_{fg}(G) = \max\left\{\max_{v \in V} \lceil d(v)/f(v) \rceil, \max_{vw \in E} \lceil p(vw)/g(vw) \rceil\right\}.$$

Then $q^*_{fg}(G) \leqq 3d_{fg}(G)/2$. ∎

Furthermore one can immediately obtain the following corollary from Theorem 13.

Corollary 15. *If all $f(v)$ and $g(vw)$ are positive even integers, then*

$$q^*_{fg}(G) = \max\left\{\max_{v \in V} \lceil d(v)/f(v) \rceil, \max_{vw \in E} \lceil p(vw)/g(vw) \rceil\right\}. \quad ∎$$

Theorem 13 is a generalization of Theorem 2 in [7], while Corollary 15 is a generalization of a corollary in [7, p. 144].

5. Application and algorithm

In this section we show that a scheduling problem on a computer network can be formulated as an *fg*-coloring of a graph, and that the proof of Theorem 4 yields an efficient approximation algorithm for the problem.

The problem is to schedule transfers of a large collection of files between various nodes of a network under port and channel constraints so as to minimize overall finishing time. In our model an instance of the problem consists of a graph $G=(V, E)$. Vertices correspond to computer centers, and edges correspond to the files to be transferred. The direction of the transfers does not matter. The integer $f(v)$ is the number of communication ports available at a computer v, and $g(vw)$ is the number of communication channels between v and w. If every file needs the same amount of time to be transferred, then our scheduling problem is formulated as a problem of finding an *fg*-coloring of G using the minimum number $q^*_{fg}(G)$ of colors. Similar scheduling problems have been discussed in [2, 11]. Note that edges colored with the same color corresponds to files that can be transferred simultaneously.

Since the ordinary edge coloring problem is NP-hard [10], the scheduling problem above is also NP-hard in general. Therefore it is unlikely that there is a polynomial-time algorithm to solve the problem exactly [1, 4]. However the proof of Theorem 4 is constructive, and immediately yields an efficient algorithm to find an *fg*-coloring of a given graph using the number of colors assured by the bound. Since any *fg*-coloring needs at least $\lceil d(v)/f(v)\rceil$ or $\lceil p(vw)/g(vw)\rceil$ colors for any vertex v or edge vw, the algorithm uses no more than twice the minimum number $q^*_{fg}(G)$ of colors. Thus the worst case ratio [4] of the algorithm is no greater than two. Using a data structure similar to that in [9], one can implement the algorithm to run in $O(|E|^2)$ time and use $O(|E|)$ space.

Goldberg's bound [5] on the ordinary chromatic index can be generalized to the case of the *f*-chromatic index [12]. We conjecture that Goldberg's and the slightly better one [13] can be further generalized to the case of *fg*-chromatic index.

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